Common Laplace Transforms Differential Equations X. Du

- All stems from $\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$
- Note: All of these formulas have conditions under which they converge!
- $\mathcal{L}[1] = \frac{1}{s}$
- $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ $\mathcal{L}[e^{at} f(t)] = F(s-a)$ Exponential *s*-shift
- $\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$ $\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$
    - Prove this easily using formulas $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$
- $\mathcal{L}[\cosh \omega t] = \frac{s}{s^2 - \omega^2}$ $\mathcal{L}[\sinh \omega t] = \frac{\omega}{s^2 - \omega^2}$
- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ Proof by induction $\mathcal{L}[t^r] = \frac{\Gamma(r+1)}{s^{r+1}}$
- $\mathcal{L}[\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$ $\mathcal{L}[\sqrt[n]{t}] = \frac{1}{s^{\frac{1}{n}+1}}\Gamma\left(\frac{1}{n}+1\right)$
- $\mathcal{L}[tf(t)] = -F'(s)$ $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$
- $\mathcal{L}[f'(t)] = sF(s) - f(0)$ $\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
    - Proof using integrate by parts
- $\mathcal{L}[f^{(n)}(t)] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$ Proof by induction
- $\mathcal{L}[f(t-a)H(t-a)] = F(s)e^{-as}$ *t*-shift $a > 0$
- $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$ *t*-scaling Proof using u-substitution
- $\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(\sigma)d\sigma$
- $\mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}dt$
    - $f(t)$ is a periodic function with period *T*. Uses geometric series and convergence.
- $\mathcal{L}[\delta(t)] = 1$ This is among the defining factors of $\delta(t)$
- $\mathcal{L}\left[f(t)g(t)\right] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)d\sigma$ *c* must lie in the region of convergence
- $\mathcal{L}\left[(f*g)(t)\right] = F(s)G(s)$ Convolution
- $\mathcal{L}\left[(H*f)(t)\right] = \mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)$